Translating Gödel's Notations

In his 1940 monograph on the consistency of the axiom of choice and the generalized continuum hypothesis, Kurt Gödel uses a number of notations which I find rather hard to remember. The following are translations from the notations that he uses in his Chapter 2 into the notations that I use in my Otter work. I mostly follow the notation used by Quaife, but on occasion my notations are different. (The notations that are used in my GOEDEL program are again slightly different.)

Gödel uses the notation $\mathfrak{Cls}(X)$ to mean that X is a class. Quaife and I assume that everything is a class, so we do not need any special notation for this notion. To express that x is a set, Gödel write $\mathfrak{M}(x)$, which Quaife and I would write as member(x,V). Gödel uses capital letters for classes and lower case letters for sets. I do not make any typographic distinction between classes and sets. Gödel writes $\mathfrak{Pr}(X)$ to denote that X is a proper class, which I write as -member(x,V).

Gödel uses the symbol $\vee$ for the inclusive or, whereas I write a solidus |, which is the standard notation used by Otter. Gödel writes a dot . for and, whereas I write &, again following the standard Otter notation.

Gödel writes $\sim$ for negation, whereas I use -.

Gödel writes XεY to denote that X belongs to Y, whereas I write member(X,Y).

Gödel leaves out the universal quantifier. So he writes (x,y) p(x,y) for what I would write as (all x y p(x,y)). Gödel uses the symbol $\exists$ for the existential quantifier, whereas I follow the Otter notation, and write exists. Gödel writes E!x to mean there exists a unique x. I do not use any abbreviation for this. If something exists and is unique, I just write out two separate clauses expressing these two facts.

Gödel uses the symbol $\supset$ for implication, whereas I write ->. He also writes $\equiv$ for logical equivalence, which I write as <->.

Gödel writes {xy} for the unordered pair, which I write as pairset(x,y), and he writes {x} for the singleton, whereas I write singleton(x).

Gödel's notation for ordered pairs is the reverse of mine. That is, his notation <xy> corresponds to what I would write as pair(y,x). Likewise, his ordered triple <xyz> = <x<yz>> corresponds to what I would write as pair(pair(z,y),x).

Gödel writes $\subseteq$ for inclusion, and $\subset$ for proper inclusion. I write subclass(x,y) for inclusion, and subclass(x,y) & -equal(x,y) for proper inclusion.

Gödel's notation $\mathfrak{Em}$(X) means that X is empty, which I would write as equal(X,0).

The notation $\mathfrak{Ex}$(X, Y) is Gödel's way of expressing that X and Y are mutually exclusive, that is, disjoint(X,Y).

Gödel writes $\mathfrak{Un}$(X) to mean that the class X is single-valued, that is,

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(all u v w (member(pair(u,v),X) & member(pair(u,w),X) -> equal(v,w))).
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This is equivalent to the statement FUNCTION(composite(Id,X)).

Gödel writes X.Y for the intersection of two classes, whereas I instead would write this out as intersection(x,y).

Gödel writes -X for the complement of a class, which I write out as complement(X).

Gödel writes $\mathfrak{D}$(X) for the domain of X, while I write D(X). He writes $\mathfrak{W}$(X) for the range, while I write R(x). He uses the term domain of values for the range.

Gödel writes the axiom of choice as follows:

$$(\exists A)\{(\mathfrak{Un}(A).(x)[\sim \mathfrak{Em}(x). \supset .(\exists y)[y\varepsilon x.<yx>\varepsilon A]]\}$$

The literal translation of this into my notation would be as follows:

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(exists z (FUNCTION(composite(Id,z)) &
(all x (-equal(x,0) -> (exists y (member(y,x) & member(pair(x,y),z))))))).
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Gödel uses the term direct product for the cartesian product, and he writes X×Y for what I would write as cart(Y,X). Note that he reverses cartesian products just as he does with ordered pairs. He also writes X^2 for cart(X,X) and X^3 for cart(cart(X,X),X).

Gödel writes $\mathfrak{Rel}$(X) to denote the fact that the class X is a relation, that is what I would write out as subclass(X,cart(V,V)). Likewise, for a ternary relation he writes $\mathfrak{Rel}_3$(X), and I would write this out as subclass(X,cart(cart(V,V),V). If A is a relation, Gödel sometimes writes xAy for $<xy>\varepsilon A$, that is, what I would write as member(pair(y,x),A).

For the inverse of a relation X Gödel uses a variety of notations, including X^{-1} and $\breve{X}$. He uses the term converse relation, and for that reason, also the notations $\mathfrak{Cnv}$(X) and $\mathfrak{Cnv}_1$(X). The only notation that I use is inverse(X).

The inverse is considered by Gödel to be a first converse. For ternary relations, Gödel also introduces what he calls a second and third converse. His notation $\mathfrak{Cnv}_2$(X) for the second converse is the same as what Quaife and I would call rotate(rotate(X)), and his third converse $\mathfrak{Cnv}_3$(X) is what Quaife and I call flip(X).

For the union of two classes, Gödel writes $X + Y$ and I write union(X,Y). He writes $X - Y$ for the relative difference, which I write out in full as intersection(X,complement(Y)).

Gödel's notation for the restriction of a relation A to a class B is $A\restriction B$, which I would write as intersection(A,cart(B,V)) or equivalently as composite(A,id(B)). Gödel writes $A\upharpoonleft B$ for what I write as intersection(A,cart(V,B)), or equivalently, composite(id(B),A).

Gödel's notation for the image is B"X, whereas I write image(B,X).

The membership relation E used by Gödel is what I write as inverse(E). The identity relation is called I by Gödel, but I denote it by Id.

Gödel denotes the composite of two relations by X|Y. This is equivalent to what I write as composite(X,Y). Note that in this case Gödel's notation is not the reverse of mine; he does not switch X and Y.

Gödel writes $\mathfrak{Un}_2$(X) when the class X and its inverse are both single-valued. That is, if FUNCTION(composite(Id,X)) & FUNCTION(inverse(X)). Gödel writes $\mathfrak{Fnc}$(X) when X is a function, which I write as FUNCTION(X).

Gödel writes A'x for what Quaife and I denote by apply(A,x).

Gödel writes X $\mathfrak{Fn}$ A to mean that X is a function with domain A. This is what I would write out as FUNCTION(X) & equal(D(X),A)).

On page 17, Gödel introduces five special functions P_1 through P_5 for which I have special notations. His P_1 is what I call SECOND, and his P_2 is what I call FIRST. Gödel writes P_3 for the function that I call SWAP. His P_4 is my inverse(ROT), and his P_5 is what I call cross(SWAP,Id).

Gödel denotes the sum class by $\mathfrak{S}(X)$. This is what Quaife and I denote by U(x). Gödel denotes the power class by $\mathfrak{P}(X)$. Quaife and I denote the power class by P(x).

The formulas 4.91–4.96 on the top of page 18 in Gödel's monograph can be translated literally as follows:

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\begin{array}{l}
\texttt{equal}(\texttt{range}(\texttt{x}), \texttt{image}(\texttt{SECOND}, \texttt{x})). \\
\texttt{equal}(\texttt{domain}(\texttt{x}), \texttt{image}(\texttt{FIRST}, \texttt{x})). \\
\texttt{equal}(\texttt{inverse}(\texttt{x}), \texttt{image}(\texttt{SWAP}, \texttt{x})). \\
\texttt{equal}(\texttt{rotate}(\texttt{rotate}(\texttt{x})), \texttt{image}(\texttt{inverse}(\texttt{ROT}), \texttt{x})). \\
\texttt{equal}(\texttt{flip}(\texttt{x}), \texttt{image}(\texttt{cross}(\texttt{SWAP}, \texttt{Id}), \texttt{x})). \\
\texttt{equal}(\texttt{cart}(\texttt{x}, \texttt{V}), \texttt{image}(\texttt{inverse}(\texttt{FIRST}), \texttt{x})).
\end{array}
$$

The fourth of these equations could be replaced by the equivalent formula

$$\texttt{equal}(\texttt{rotate}(\texttt{x}), \texttt{image}(\texttt{ROT}, \texttt{x})),$$

while the fifth equation can also be rewritten in the equivalent form

$$\texttt{equal}(\texttt{flip}(\texttt{x}), \texttt{composite}(\texttt{x}, \texttt{SWAP})).$$